

Asymmetric Homomorphism Thresholds for Graphs of Large Odd Girth

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Abstract

We determine the asymmetric homomorphism threshold from graphs of odd girth at least 7 to triangle-free graphs, showing that $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) = \frac{1}{9}$. Equivalently, for every $\varepsilon > 0$, every n -vertex graph of odd girth at least 7 and minimum degree at least $(1/9 + \varepsilon)n$ admits a homomorphism to a triangle-free graph of size bounded by a function of ε , while there exist graphs of odd girth at least 7 and minimum degree at least $(1/9 - \varepsilon)n$ for which no such bounded-size triangle-free homomorphic image exists.

More generally, for every $t \geq 3$, we prove $\delta_{\text{hom}}(\{C_3, C_5\}, \{K_t\}) = \frac{1}{3t}$. In particular, for every proper monotone class $\mathcal{C}$ of graphs, the threshold for graphs of odd girth at least 7 to admit a homomorphism to a bounded-size graph in $\mathcal{C}$ is positive. We further extend this phenomenon to arbitrary odd girth: for every $k \geq 2$ and every proper monotone subclass $\mathcal{C}$ of the class of graphs of odd girth at least $2k - 1$, the threshold for graphs of odd girth at least $2k + 3$ to admit a homomorphism to a bounded-size graph in $\mathcal{C}$ is positive.

These results disprove conjectures of Gishboliner, Hurley and Wigderson and exhibit a sharp contrast with the corresponding zero chromatic-threshold results. Our lower-bound constructions are based on high-dimensional Borsuk graphs, while the matching upper bounds use regularity arguments to recover the structure underlying these constructions.

1 Introduction

A central theme in extremal combinatorics is to study how the imposition of local constraints on a discrete object (e.g. forbidding a constant-sized subgraph) can yield consequences on the global structure. Many results in this direction state that if we impose a local constraint on a graph, then the global structure must become increasingly rigid as we insist on having more and more edges. For example, consider the constraint of having no triangles. For graphs with very few edges, this imposes essentially no constraint on the global structure. However, as the number of edges increases, the structure becomes more rigid; for example, an elegant result of Andrásfai [And62] states that if the minimum degree of a triangle-free graph G is more than $\frac{2}{5}|V(G)|$, then G must be bipartite, i.e. have a very rigid global structure. Moreover, once the number of edges increases beyond $\frac{1}{4}|V(G)|^2$, the structure becomes even more rigid—such a graph simply does not exist [Man07]. We remark that several recent breakthroughs in Ramsey theory and graph theory rely on the discovery that there exist highly unstructured triangle-free (or, more generally, K_s -free) graphs with substantially more edges than naive probabilistic reasoning would suggest [Bra26, CJMS25, HHKP25, KSSW25, MSV26].

In order to discuss this structural rigidity further, we introduce the following definition. Given a family $\mathcal{F}$ of graphs, its *chromatic threshold* $\delta_{\chi}(\mathcal{F})$ is the infimum of all $c \in [0, 1]$ for which there exists $C \in \mathbb{N}$ such that every n -vertex $\mathcal{F}$ -free (i.e. which does not contain any $F \in \mathcal{F}$ as a (not necessarily induced) subgraph) graph with minimum degree at least cn has chromatic number at most C . Thus, for

example, the result of Andrásfai mentioned above shows that $\delta_\chi(C_3) \leq 2/5$. A construction of Hajnal (see [ES73]) shows that $\delta_\chi(C_3) \geq 1/3$. Nearly 30 years later, Thomassen [Tho02] proved a matching upper bound, thereby establishing that $\delta_\chi(C_3) = 1/3$. Brandt and Thomassé [BT04] strengthened Thomassen’s result by showing that n -vertex triangle-free graphs with minimum degree greater than $n/3$ have chromatic number at most 4. Thomassen [Tho07] then proved that $\delta_\chi(C_5) = 0$, which in particular implies $\delta_\chi(\{C_3, C_5\}) = 0$. Said differently, the chromatic threshold of graphs with odd girth¹ at least 7 is 0. To see it, let G be a graph with odd girth at least 7 and minimum degree cn for some $c > 0$. Let $\{v_1, \dots, v_k\} \subseteq V(G)$ be maximal such that $N(v_i) \cap N(v_j) = \emptyset$ for all $i \neq j$. Since G has minimum degree at least cn , it follows that $k \leq 1/c$. Since the set $\{v_1, \dots, v_k\}$ is maximal, every $v \in V(G)$ is at distance at most two from some v_i . However, since G contains no C_3 and no C_5 , $N(v_i)$ and $N^2(v_i) \cup \{v_i\}$ are independent sets for every $1 \leq i \leq k$, which together cover the full vertex set of G . Thus, G has chromatic number at most $2k \leq 2/c$.

While the chromatic threshold captures one way of formalizing the structural rigidity of dense $\mathcal{F}$ -free graphs, we would often like to understand their structure more precisely. Indeed, while having bounded chromatic number is a natural way of insisting that a graph be “highly structured”, it is natural to expect that dense $\mathcal{F}$ -free graphs should have some additional structure reflecting their $\mathcal{F}$ -freeness. For instance, in the result of Andrásfai discussed above, we see that a sufficiently dense triangle-free graph is bipartite, and of course all bipartite graphs are triangle-free; hence, the structural rigidity also provides a *witness* for the $\mathcal{F}$ -freeness in this case.

To formalize this idea, we recall the following definition. The *homomorphism threshold* $\delta_{\text{hom}}(\mathcal{F})$ of $\mathcal{F}$ is the infimum of all $c \in [0, 1]$ such that there exists a finite $\mathcal{F}$ -free graph H_c such that every n -vertex $\mathcal{F}$ -free graph with minimum degree at least cn admits a homomorphism to H_c . Equivalently², $\delta_{\text{hom}}(\mathcal{F})$ is the infimum of all c such that every n -vertex $\mathcal{F}$ -free graph with minimum degree at least cn admits a homomorphism to an $\mathcal{F}$ -free graph of bounded size, where the bound depends only on c . It is immediate from the definition that $\delta_\chi(\mathcal{F}) \leq \delta_{\text{hom}}(\mathcal{F})$. These two values sometimes coincide: Łuczak [Lu06] proved that $\delta_{\text{hom}}(C_3) = 1/3$. However, they are not always equal, for instance $\delta_{\text{hom}}(\{C_3, C_5\}) = 1/5$, as proved by Letzter and Snyder [LS18]. The lower bound follows from a construction of Andrásfai [And62, And64], see also [Erd57]: Consider the graph on the unit circle $\mathbb{S}^1$ where two vertices u, v are adjacent whenever the angle between them is greater than $4\pi/5$. This graph is $\{C_3, C_5\}$ -free and contains n -vertex induced subgraphs with minimum degree arbitrarily close to $n/5$ with no homomorphism to any fixed $\{C_3, C_5\}$ -free graph. We remark that there have recently been a number of exciting new results on homomorphism thresholds, e.g. [HRS26, GHL26, HLZH26, San22, WWX26].

A natural relaxation of the homomorphism threshold is to require the existence of a homomorphism to a graph H that is not $\mathcal{F}$ -free, but $\mathcal{F}'$ -free, for some $\mathcal{F}' \neq \mathcal{F}$. Following the notation of Gishboliner, Hurley and Wigderson [GHW26], given two families $\mathcal{F}_1, \mathcal{F}_2$ of graphs, their *asymmetric homomorphism threshold* $\delta_{\text{hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is the infimum of all $c \in [0, 1]$ such that there exists a finite $\mathcal{F}_2$ -free graph H_c such that every n -vertex $\mathcal{F}_1$ -free graph with minimum degree at least cn admits a homomorphism to H_c . In [GHW26], they show that $\delta_{\text{hom}}(\{C_3, C_5, C_7\}, \{C_3\}) = 0$, which contrasts with the result of Ebsen and Schacht [ES20] that $\delta_{\text{hom}}(\{C_3, C_5, C_7\}) = 1/7$. This comparison isolates what the positive symmetric threshold measures: the obstruction is not to bounded homomorphic compression itself, but to compression that preserves the full odd-girth condition. Indeed, every positive minimum-degree ratio forces a bounded triangle-free image, whereas below $1/7$ one cannot in general choose this image to avoid both C_5 and C_7 . This naturally raises the question of how much of the forbidden-cycle structure of the source can always be retained in a bounded image. In this direction, Gishboliner, Hurley and Wigderson conjectured that $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) = 0$, which would strengthen Thomassen’s result that $\delta_\chi(\{C_3, C_5\}) = 0$ and pinpoint the complexity of the homomorphism threshold for $\{C_3, C_5\}$ -free graphs. Our first main result is to disprove this conjecture, by precisely determining $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\})$ as an absolute positive constant.

Theorem 1.1. $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) = \frac{1}{9}$.

Observe that $\delta_\chi(\{C_3, C_5\}) = 0$ is equivalent to $\delta_{\text{hom}}(\{C_3, C_5\}, \emptyset) = 0$. It is then natural to wonder whether this statement can be strengthened at all, namely whether there exists a proper monotone³

¹The *odd girth* of a graph G , denoted $og(G)$, is the length of a shortest odd cycle in G (or ∞ if G contains no odd cycle).

²When every graph in $\mathcal{F}$ is connected, as is the case in the paper.

³A class of graphs is *monotone* if it is closed under taking subgraphs. When an ambient monotone class $\mathcal{G}$ is specified, a monotone subclass $\mathcal{C} \subseteq \mathcal{G}$ is *proper* if $\mathcal{C} \neq \mathcal{G}$.

class $\mathcal{C}$ of graphs such that the threshold for $\{C_3, C_5\}$ -free graphs to have a homomorphism to a bounded-size graph in $\mathcal{C}$ is 0. We show that this is not the case.

Theorem 1.2. *For every $t \geq 3$, we have $\delta_{\text{hom}}(\{C_3, C_5\}, \{K_t\}) = \frac{1}{3t}$. Therefore, for every graph H , we have $\delta_{\text{hom}}(\{C_3, C_5\}, \{H\}) > 0$.*

Theorem 1.2 showcases a surprising contrast between the threshold for the existence of homomorphisms to bounded-size graphs and the threshold for the existence of homomorphisms to bounded-size graphs from a proper monotone class for $\{C_3, C_5\}$ -free graphs. Indeed, for $\{C_3, C_5\}$ -free graphs, there is a sharp qualitative change between merely forcing a bounded homomorphic image and forcing such an image to satisfy any fixed nontrivial monotone restriction.

The result of Thomassen was extended by Gishboliner, Hurley and Wigderson [GHW26], who showed that for every $k \geq 2$, $\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}\}) = 0$. They conjectured that this result could be strengthened to $\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, C_{2k-1}\}) = 0$. Again, it is natural to relax their statement to ask whether there exists a proper monotone subclass $\mathcal{C}$ of the class of graphs of odd girth at least $2k - 1$ such that the threshold for graphs of odd girth at least $2k + 3$ to have a homomorphism to a bounded-size graph in $\mathcal{C}$ is 0. We also refute this more general statement.

Theorem 1.3. *For every $k \geq 2$, for every graph H of odd girth at least $2k - 1$, we have*

$$\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, H\}) > 0.$$

In Theorem 1.2, we are able to pinpoint the actual threshold for cliques. We then derive the result using that the family of cliques is *universal*: every graph H is a subgraph of some clique. For $k \geq 2$, the class of $(k - 2)$ -fold Mycielskians is a class of graphs of odd girth at least $2k - 1$ that is universal for the class of graphs of odd girth at least $2k - 1$. To prove Theorem 1.3, for every $k \geq 2$ and every $(k - 2)$ -fold Mycielskian Γ , we in fact establish that $\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, \Gamma\}) = \frac{1}{3|\Gamma|}$, and then immediately derive Theorem 1.3 from there.

1.1 Proof sketch of the lower bound

In our opinion, the most surprising finding in this paper is the lower bound in Theorem 1.1, and specifically the counterexample to the conjecture of [GHW26] that $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) = 0$. As such, we end this introduction by sketching a proof of the lower bound $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) \geq \frac{1}{9}$; the proof of the more general Theorem 1.3 follows the same approach, albeit with more technical complications to obtain the general result. We remark that the proof of the matching upper bound $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) \leq \frac{1}{9}$ is also inspired by the structure of our lower-bound construction, and the proof of the upper bound essentially “uncovers” this structure. Thus, while this is by no means the unique construction witnessing $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\}) \geq \frac{1}{9}$, it is the case that all such constructions must be of a similar form.

The key object in the construction is the *Borsuk graph* $\text{Bor}(d, \theta)$. This is an infinite graph, whose vertex set is the d -dimensional unit sphere $\mathbb{S}^d$, and in which two vertices u, v are adjacent if and only if $\text{dist}(u, v) \geq \pi - \theta$, where dist denotes the angular distance between u and v . The main property we require of the Borsuk graph (for now) is that $\chi(\text{Bor}(d, \theta)) \geq d + 2$ for any d, θ ; this is a simple but important consequence of the Borsuk–Ulam theorem [Bor33] (and the bound $d + 2$ is optimal for all sufficiently small θ). Although $\text{Bor}(d, \theta)$ is an infinite graph, one can approximate it by finite graphs by taking the induced subgraph on a sufficiently fine ε -net; in this high-level proof overview we do not dwell on this point.

Before presenting our construction, let us first present the construction of Hajnal showing that $\delta_\chi(C_3) \geq \frac{1}{3}$. More precisely, we will discuss a variant of Hajnal’s construction due to Łuczak and Thomassé [LT10], which is based on Borsuk graphs. We define a graph G_0 on vertex parts X, Y, Z as follows. First, X is a copy of $\text{Bor}(d, \theta)$ for some large d and small θ , to be chosen later, while Y and Z are independent sets. Between Y and Z we put a complete bipartite graph, while between X and Y we put a “hemisphere graph”: we identify the vertices of Y with $\mathbb{S}^d$ as well, and we join $x \in X$ to $y \in Y$ if and only if $\text{dist}(x, y) \leq \frac{\pi}{2} - \theta$. That is, for any $y \in Y$, its neighborhood in X is essentially the hemisphere centered at y ; more precisely, it is the spherical cap centered at y which is θ -far from being a full hemisphere.

Since $Bor(d, \theta)$ is an induced subgraph of G_0 , we see that $\chi(G_0) \geq d + 2$. Hence, by picking d appropriately, we can ensure that $\chi(G_0)$ is arbitrarily large. Moreover, it is not hard to check that G_0 is triangle-free for sufficiently small θ : the key point here is that if $x, x' \in X$ are both adjacent to some $y \in Y$, then they both lie in the spherical cap around y , and hence their distance must be at most $\pi - 2\theta$ by the triangle inequality, and thus they are non-adjacent in $Bor(d, \theta) \cong G_0[X]$. Note too that, as $\theta \rightarrow 0$, the edge density between X and Y tends to $\frac{1}{2}$, since the spherical cap approaches a hemisphere and hence picks up almost half the mass of the sphere. The final step is now to duplicate some vertices of G_0 in order to obtain the desired minimum degree. Namely, we duplicate vertices appropriately to ensure that roughly one-third of the vertices lie in Z and roughly two-thirds of the vertices lie in Y (and X has a negligible fraction of the vertices). In this way, every vertex is adjacent to almost one-third of the other vertices, completing the proof that $\delta_\chi(C_3) \geq \frac{1}{3}$.

Our construction for the lower bound on $\delta_{\text{hom}}(\{C_3, C_5\}, \{C_3\})$ can be viewed as “gluing together” three copies of G_0 . Namely, let us define a graph G on vertex sets $X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3$. As before, we make each (Y_i, Z_i) complete bipartite, and each (X_i, Y_i) a “hemisphere graph”. The main twist is that now each X_i is also an independent set, rather than a Borsuk graph. Instead, we place the tripartite cover of the Borsuk graph on the vertices $X_1 \cup X_2 \cup X_3$. That is, for all $i \neq j$ and all $x \in X_i, x' \in X_j$, we make xx' an edge of G if and only if $\text{dist}(x, x') \geq \pi - \theta$. By duplicating vertices as above, we can ensure that the minimum degree of G is arbitrarily close to $\frac{1}{9}|V(G)|$ (just because the minimum degree in each $X_i \cup Y_i \cup Z_i$ is one-third, and there are three such parts). It is also not hard to check that G is $\{C_3, C_5\}$ -free; the key point now is that for any $y \in Y_1$ (say), any two of its neighbors in X_1 cannot be joined by a path of length 3, as any two vertices in X_1 at distance 3 must still be nearly antipodal, and no two elements of a spherical cap are nearly antipodal.

So it remains to prove that G has no bounded-size triangle-free homomorphic image. For contradiction, suppose that $\varphi : G \rightarrow H$ is a homomorphism, where H is some triangle-free graph. Let $x, x' \in X_1$ with $\varphi(x) = \varphi(x')$. We note that x, x' cannot be joined by a path of length 3 in G , for if they were, the image of this path under φ would yield a triangle in H . Hence, if $\varphi(x) = \varphi(x')$, then $\text{dist}(x, x') \leq \pi - 3\theta$. That is, φ defines a proper coloring of $Bor(d, 3\theta)$ with $|H|$ colors, and hence $|H| \geq \chi(Bor(d, 3\theta)) \geq d + 2$. Thus, by picking d appropriately, we can rule out triangle-free homomorphic images of G of any fixed size, as claimed.

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AI Disclosure. We used assistance from ChatGPT 5.5 Pro, which helped us discover the proof for the upper bound in Theorem 1.1. All remaining proofs and results are due to the authors. Moreover, this paper, including all its proofs, is fully written by the (human) authors.

2 Non-existence of homomorphisms

In this section, we prove the lower bounds of Theorem 1.2 and Theorem 1.3. For this, we need some preparation.

2.1 Generalized Mycielskians

For an integer n , we denote by $[n]$ the set $\{1, \dots, n\}$. Given an integer $k \geq 0$ and a graph Γ , its k -fold Mycielskian is the graph $M_k(\Gamma)$ defined as follows:

1. $V(M_k(\Gamma)) = (V(\Gamma) \times [k + 1]) \cup \{r\}$. That is, the vertices of $M_k(\Gamma)$ are of the form (v, i) for $v \in V(\Gamma)$ and $i \in [k + 1]$, as well as a single fresh vertex r .
2. The induced subgraph on $V(\Gamma) \times \{k + 1\}$ is a copy of Γ and $V(\Gamma) \times \{i\}$ is an independent set for all $i \in [k]$.
3. r is adjacent to all vertices in $V(\Gamma) \times \{1\}$ and no other vertices.
4. For all $i \in [k]$, each vertex (v, i) is adjacent to all other vertices $(w, i + 1)$ such that $vw \in E(\Gamma)$ and to no other vertices of $V(\Gamma) \times \{i + 1\}$.

The sequence of k -fold Mycielskians is the sequence $(\Gamma_k^t)_{t \geq 2}$ where $\Gamma_k^2 = K_2$ and $\Gamma_k^{t+1} = M_k(\Gamma_k^t)$ for every $t \geq 2$.

We now review some basic properties of $(\Gamma_k^t)_{t \geq 2}$. The first one was proved by Stiebitz [Sti85] when he introduced generalized Mycielskians. To the best of our knowledge, the second was first proved by Cropper, Gyárfás and Lehel [CGL06] for Mycielskians. For completeness, we include a proof of the result for generalized Mycielskians.

Lemma 2.1 ([Sti85]). *Let $k \geq 0$ and let G be a graph. If G has odd girth $2k + 3$, so does $M_k(G)$. In particular, for every $t \geq 3$, Γ_k^t has odd girth $2k + 3$.*

Lemma 2.2. *Let $k \geq 0$ and let G be a graph of odd girth greater than $2k + 1$. Then, there exists $t \geq 2$ such that G is a subgraph of Γ_k^t .*

Proof. We prove by induction on the number of vertices of G that G is a subgraph of $\Gamma_k^{|V(G)|}$. The statement is trivial if $|V(G)| = 2$. Suppose now that the statement holds for all graphs on some number n of vertices, and let a graph G with $|V(G)| = n + 1$ and odd girth greater than $2k + 1$. Let us pick any $v \in V(G)$. Then $G - v$ is a subgraph of Γ_k^n by the induction hypothesis so there exists an embedding $\psi : V(G - v) \rightarrow V(\Gamma_k^n)$ that witnesses it. We define an embedding $\phi : V(G) \rightarrow V(\Gamma_k^{n+1})$ as follows.

- $\phi(v) = r$.
- For every $u \in V(G) \setminus \{v\}$, if $\text{dist}_G(v, u) \leq k$, then define $\phi(u) := (\psi(u), \text{dist}_G(v, u))$.
- Otherwise, define $\phi(u) := (\psi(u), k + 1)$.

Since G has odd girth greater than $2k + 1$, it follows that for every $i \in [k]$, the set of vertices at distance i from v is an independent set in G . It is then straightforward to verify that ϕ witnesses that G is a subgraph of Γ_k^{n+1} . $\square$

The next lemma is a simple observation about the existence of walks of prescribed length between different vertices of generalized Mycielskians.

Lemma 2.3. *Let $k \geq 1$ and let Γ be a graph. Let $u, v \in V(\Gamma)$ such that there is a walk of length ℓ between u and v in Γ . Let $i \leq j \in [k + 1]$. There exists a walk of length ℓ in $M_k(\Gamma)$ between (u, i) and (v, j) if one of the following holds:*

- ℓ and $j - i$ have the same parity and $\ell \geq j - i$, or
- $\ell \geq 2k + 2 - (i + j)$.

Proof. Let $u = u_0 u_1 \dots u_\ell = v$ be a walk of length ℓ between u and v in Γ . Suppose first that ℓ and $j - i$ have the same parity and $\ell \geq j - i$. Since $k \geq 1$, there exists $j' \in \{j - 1, j + 1\} \cap [k + 1]$. Then, $(u, i)(u_1, i + 1) \dots (u_{j-i}, j)(u_{j-i+1}, j')(u_{j-i+2}, j) \dots (u_{\ell-1}, j')(v, j)$ is a walk of length ℓ in $M_k(\Gamma)$ between (u, i) and (v, j) . To see that this walk is well-defined, we use that $\ell \geq j - i$, and that ℓ and $j - i$ have the same parity (so that after the $\ell - (j - i)$ last steps we are indeed back at (v, j) and not at (v, j')).

Suppose now that $\ell \geq 2k + 2 - (i + j)$. Then, $(u, i)(u_1, i + 1) \dots (u_{k+1-i}, k + 1)(u_{k+2-i}, k + 1) \dots (u_{j+\ell-k-1}, k + 1)(u_{j+\ell-k}, k) \dots (u_{\ell-1}, j + 1)(v, j)$ is a walk of length ℓ in $M_k(\Gamma)$ between (u, i) and (v, j) . To see that this walk is well-defined, we need $k + 1 - i \leq j + \ell - k - 1$, i.e. $\ell \geq 2k + 2 - (i + j)$. $\square$

Finally, we need the following lemma, which shows that generalized Mycielskians are “rich” in closed walks of length $2k + 3$.

Lemma 2.4. *Let $k \geq 0$ and let Γ be a graph such that any two vertices of Γ belong to a closed walk of length $2k + 3$. Then any two vertices of $M_k(\Gamma)$ belong to a closed walk of length $2k + 3$.*

Proof. Since any two vertices of Γ belong to a closed walk of length $2k + 3$, Γ must contain an edge. For $k = 0$, observe that $M_k(\Gamma)$ is the graph obtained from Γ by adding a universal vertex. It is then straightforward to check that any two vertices of $M_k(\Gamma)$ belong to a closed walk of length 3.

Suppose now that $k \geq 1$. Recall that $V(M_k(\Gamma)) = (V(\Gamma) \times [k + 1]) \cup \{r\}$. First, we prove that for every $(v, i) \in V(\Gamma) \times [k + 1]$, (v, i) and r belong to a closed walk of length $2k + 3$ in $M_k(\Gamma)$. By assumption, v belongs to a closed walk of length $2k + 3$ in Γ , so v has a neighbor u in Γ . Then,

$(u, k+1)(v, k)(u, k-1) \dots r \dots (v, k-1)(u, k)(v, k+1)(u, k+1)$ is a closed walk of length $2k+3$ in $M_k(\Gamma)$ which visits (v, i) and r . Note that this also shows that there is a walk of length i and a walk of length $2k+3-i$ between (v, i) and r . It also proves that for every $v \in V(\Gamma)$ and $i \neq j \in [k+1]$, (v, i) and (v, j) belong to a closed walk of length $2k+3$.

Finally, let $u \neq v \in V(\Gamma)$ and $i, j \in [k+1]$. By symmetry, we can assume that $i \leq j$. Suppose first that there is a walk of length $j-i$ between u and v in Γ . By Lemma 2.3, there is a walk of length $j-i$ in $M_k(\Gamma)$ between (u, i) and (v, j) . Furthermore, there is a walk of length i between (u, i) and r , and a walk of length $2k+3-j$ between (v, j) and r . The concatenation of these three walks is a closed walk of length $2k+3$ that visits (u, i) and (v, j) . Suppose now that there is a walk of length $2k+3-i-j$ between u and v in Γ . By Lemma 2.3, there is a walk of length $2k+3-i-j$ in $M_k(\Gamma)$ between (u, i) and (v, j) . Furthermore, there is a walk of length i between (u, i) and r , and a walk of length j between (v, j) and r . The concatenation of these three walks is a closed walk of length $2k+3$ that visits (u, i) and (v, j) . Finally, suppose that we are not in one of those two cases. By assumption, there is a closed walk W of length $2k+3$ in Γ that goes through u and v . Write $W = W_1 W_2$ such that W_1 and W_2 are both walks with endpoints u and v . By symmetry, we can assume that the length of W_1 , call it d , has the same parity as $j-i$. Note that W_2 has length $2k+3-d$. Since there is no walk of length $j-i$ in Γ between u and v in Γ and d and $j-i$ have the same parity, we have $d > j-i$. By Lemma 2.3, there is a walk of length d between (u, i) and (v, j) in $M_k(\Gamma)$. Since there is no walk of length $2k+3-i-j$ between u and v in Γ and $2k+3-d$ and $2k+3-i-j$ have the same parity, we have $2k+3-d > 2k+3-i-j$. By Lemma 2.3, there is a walk of length $2k+3-d$ between (u, i) and (v, j) in $M_k(\Gamma)$. The concatenation of these two walks is a closed walk of length $2k+3$ that visits (u, i) and (v, j) . $\square$

Throughout the paper, we will implicitly use the following fact.

Observation 2.5. *Let G be a graph that contains a closed odd walk of length at most ℓ . Then G contains an odd cycle of length at most ℓ .*

As a corollary of this observation and Lemma 2.4, we immediately obtain the following.

Lemma 2.6. *Let $k \geq 0$ and $t \geq 3$. Then any two vertices of Γ_k^t belong to a cycle of length $2k+3$.*

Proof. We prove it by induction on t , with $k \geq 0$ fixed. For $t = 3$, observe that $\Gamma_k^t = M_k(K_2) = C_{2k+3}$ so the property holds. Let $t \geq 3$ such that the property holds. Since $\Gamma_k^{t+1} = M_k(\Gamma_k^t)$, Lemma 2.4 implies that any two vertices of Γ_k^{t+1} belong to a closed walk of length $2k+3$. Since Γ_k^{t+1} has odd girth at least $2k+3$ by Lemma 2.1, this closed walk must be a cycle, which proves the desired result. $\square$

Finally, the following lemma places restrictions on homomorphic images of generalized Mycielskians: every homomorphic image of Γ_k^t with high odd girth must actually contain Γ_k^t as an induced subgraph.

Lemma 2.7. *Let $k \geq 0$ and $t \geq 3$ and let J be a graph of odd girth at least $2k+3$ such that there exists a homomorphism from Γ_k^t to J . Then J contains Γ_k^t as an induced subgraph.*

Proof. Let $\phi : V(\Gamma_k^t) \rightarrow V(J)$ be a homomorphism. First, we show that ϕ is injective. By contradiction, suppose not: there exist $x \neq y \in V(\Gamma_k^t)$ such that $\phi(x) = \phi(y)$. By Lemma 2.6, there exists a path P between x and y in Γ_k^t whose length is odd and smaller than $2k+3$. Since ϕ is a homomorphism, the image of P under ϕ is a walk of the same length between $\phi(x)$ and $\phi(y)$. Thus, there is a closed walk in J whose length is odd and smaller than $2k+3$. This contradicts the fact that J has odd girth at least $2k+3$. Therefore, J contains Γ_k^t as a subgraph. By Lemma 2.6, any two vertices of Γ_k^t belong to a cycle of length $2k+3$, so Γ_k^t is edge-maximal with the property of having odd girth at least $2k+3$. Since J has odd girth at least $2k+3$, this implies that the copy of Γ_k^t in J is induced, as desired. $\square$

2.2 High-dimensional spheres and Borsuk graphs

We denote by $\mathbb{S}^d$ the unit sphere of $\mathbb{R}^{d+1}$. Given $x \in \mathbb{S}^d$, its *antipodal* point is $-x$. We consider the distance $\text{dist} : \mathbb{S}^d \times \mathbb{S}^d \rightarrow \mathbb{R}^+$ such that for all $u, v \in \mathbb{S}^d$, $\text{dist}(u, v) \in [0, \pi]$ is the angular distance between u and v . For instance, antipodal points are at distance π while orthogonal points are at distance $\pi/2$.

We recall the construction of the (infinite) d -dimensional θ -Borsuk graph $\text{Bor}(d, \theta)$ whose vertex set consists of the unit sphere $\mathbb{S}^d$ and where $u, v \in \mathbb{S}^d$ are adjacent if $\text{dist}(u, v) \geq \pi - \theta$. It is well-known

(cf. [EH67]) that the odd girth of $Bor(d, \theta)$ is arbitrarily large when $\theta > 0$ is sufficiently small and that the fractional chromatic number⁴ of $Bor(d, \theta)$ tends to 2 as θ approaches 0.

Arguments concerning the complexity of the d -dimensional θ -Borsuk graph typically rely on the famous *Borsuk–Ulam theorem* [Bor33], which can be equivalently cast as the fact that $Bor(d, \theta)$ has chromatic number at least $d + 2$. See [Mat03] for an overview of the applications of the Borsuk–Ulam theorem in combinatorics.

2.3 A general construction

The following proposition is a generalization of the construction sketched in the introduction. Rather than simply placing a tripartite Borsuk graph on the parts X_1, X_2, X_3 as we did there, we instead use a given graph H as a “template” for how to place Borsuk graphs; in the example in the introduction, this template graph H would be a triangle.

Proposition 2.8. *For every integer $N \geq 1$, for every $\varepsilon > 0$, and for every graph H which contains an odd cycle, there exists a graph H' with the following properties:*

- $og(H') \geq og(H) + 4$,
- $\delta(H') \geq \left(\frac{1}{3|V(H)|} - \varepsilon\right) |V(H')|$,
- If H' has a homomorphism to a graph J on at most N vertices then H also has a homomorphism to J .

Proof. Let $d \in \mathbb{N}$ be large enough so that for every $\theta > 0$, the Borsuk graph $Bor(d, \theta)$ has chromatic number greater than $N^{|V(H)|}$. Define $\zeta = \varepsilon |V(H)|$ and let $\eta > 0$ be chosen small enough so that $\frac{1}{2+\eta} \cdot \frac{1}{3/2+\eta} \geq \frac{1}{3} - \zeta$. Let $\theta \in (0, \pi)$ be small enough so that $\chi_f(Bor(d, \theta)) \leq 2 + \eta$ and let $\iota < \frac{\theta}{og(H)+2}$. By the *de Bruijn–Erdős theorem*, there exists a finite set $X \subseteq \mathbb{S}^d$ such that $\chi(Bor(d, \iota)[X]) = \chi(Bor(d, \iota)) > N^{|V(H)|}$. Let $F := Bor(d, \theta)[X]$, and observe that $\chi_f(F) \leq \chi_f(Bor(d, \theta)) \leq 2 + \eta$.

Let $\mathcal{I}$ be the set of all independent sets of F . Since $\chi_f(F) \leq 2 + \eta$, there exists a weight function $w : \mathcal{I} \rightarrow \mathbb{R}^+$ such that $\sum_{v \in I \in \mathcal{I}} w(I) \geq 1$ for every $v \in X$ and $w(\mathcal{I}) := \sum_{I \in \mathcal{I}} w(I) \leq 2 + \eta$. Since $\chi_f(F)$ is the solution of a (finite) linear program with integral coefficients, we may assume without loss of generality that $w : \mathcal{I} \rightarrow \mathbb{Q}^+$. Thus, after rescaling, there exists a function $w' : \mathcal{I} \rightarrow \mathbb{N}$ such that $\sum_{v \in I \in \mathcal{I}} w'(I) \geq \frac{1}{2+\eta} \cdot w'(\mathcal{I})$ for every $v \in X$. By considering the (bipartite) incidence graph between the vertices X and the independent sets $\mathcal{I}$, and replacing each vertex corresponding to an independent set $I \in \mathcal{I}$ by $w'(I)$ copies of itself, it follows that there exists a set Y and a bipartite graph B with bipartition (X, Y) such that $|N_B(v)| \geq \frac{1}{2+\eta} \cdot |Y|$ for every $v \in X$. Note that Y can be chosen to have even size and arbitrarily large, by replacing each element of Y by a large (but fixed) number of copies of itself. In particular, we can assume that $|Y|$ is even and that $|X| \leq \eta |Y|$. By adding a set Z of size $|Y|/2$ that is complete to Y , we obtain a bipartite graph G with bipartition $(X \cup Z, Y)$. Note that $|V(G)| = |X| + |Z| + |Y| = |X| + \frac{3}{2}|Y| \leq (\frac{3}{2} + \eta) |Y|$. Then, every $v \in X \cup Z$ satisfies $|N_G(v)| \geq \frac{1}{2+\eta} \cdot |Y| \geq \frac{1}{2+\eta} \cdot \frac{1}{3/2+\eta} |V(G)| \geq (\frac{1}{3} - \zeta) |V(G)|$ by definition of η . Also, every $v \in Y$ satisfies $|N_G(v)| \geq |Z| = \frac{|Y|}{2} \geq \frac{1}{2} \cdot \frac{|V(G)|}{3/2+\eta} \geq \frac{1}{2+\eta} \cdot \frac{1}{3/2+\eta} |V(G)| \geq (\frac{1}{3} - \zeta) |V(G)|$ again by definition of η . Thus, $\delta(G) \geq (\frac{1}{3} - \zeta) |V(G)|$.

For every $h \in V(H)$, let G_h be a copy of G such that all copies G_h are vertex-disjoint. For every $h \in V(H)$, denote the vertex set of G_h by $X_h \cup Y_h \cup Z_h$, and for every $x \in X$ let x_h denote the copy of x in X_h . Then, let H' be the graph obtained from the disjoint union of all G_h by adding edges between the sets X_h as in the tensor (or categorical) product $Bor(d, \iota) \times H$. Formally, connect x_h and x'_h for $x, x' \in X$ and $h, h' \in V(H)$ if and only if $xx' \in E(Bor(d, \iota))$ (i.e. if $\text{dist}(x, x') \geq \pi - \iota$) and $hh' \in E(H)$.

We now show that H' satisfies the desired properties. By contradiction, suppose that $og(H') < og(H) + 4$, so $og(H') \leq og(H) + 2$. Since we can always extend the length of a closed walk by any even number by moving forth and back along some edge, it follows that there exists a closed walk W' of length *exactly* $\ell := og(H) + 2$ in H' . Let H^+ be the graph obtained from H by adding for every

⁴The fractional chromatic number $\chi_f(G)$ of a graph G is the infimum of $\sum_{I \in \mathcal{I}(G)} w_I$ over all choices of nonnegative weights $(w_I)_{I \in \mathcal{I}(G)}$ on the independent sets of G such that, for every vertex $v \in V(G)$, we have $\sum_{I \ni v} w_I \geq 1$.

$h \in V(H)$ a disjoint edge $e_h := a_h b_h$ which we connect to h via another edge ha_h . By construction, the map $\phi : V(H') \rightarrow V(H^+)$ that maps each set X_h to h , each set Y_h to a_h and each set Z_h to b_h , is a homomorphism from H' to H^+ . Thus, ϕ maps W' to a closed walk W^+ of length ℓ in H^+ . Suppose first that W^+ only visits vertices of H in H^+ . Then, we can write $W' = x_{h_0}^0 x_{h_1}^1 \dots x_{h_{\ell-1}}^{\ell-1} x_{h_0}^0$ where $h_i \in V(H)$ and $x_i \in X$ for every $i \in \{0, \dots, \ell\}$. A simple proof by induction shows that $\text{dist}(x^0, x^j) \leq j\iota$ for every even $j \in \{0, \dots, \ell-1\}$. Since $\ell-1$ is even, we have $\text{dist}(x^0, x^{\ell-1}) \leq (\ell-1)\iota$. However, since $x_{h_{\ell-1}}^{\ell-1} x_{h_0}^0$ is an edge of H' , we have $\text{dist}(x^0, x^{\ell-1}) \geq \pi - \iota$, so $\pi - \iota \leq (\ell-1)\iota$ so $\iota \geq \frac{\pi}{\ell} = \frac{\pi}{og(H)+2}$, a contradiction since $\iota < \frac{\theta}{og(H)+2} < \frac{\pi}{og(H)+2}$ by definition. Thus, W^+ does not only visit vertices from $V(H)$ in H^+ . Since W^+ has length $og(H) + 2$ and every odd cycle of H^+ lives in $V(H)$, so has length at least $og(H)$, it follows that there exists $h \in V(H)$ so that W^+ visits h , then a_h , then h again, and then forms a cycle of length $og(H)$ that only visits vertices of H in H^+ . Let x_h^0 and x_h^2 be the two vertices visited by W' that belong to X_h . Then, x_h^0 and x_h^2 have a common neighbor in Y_h , and so there is an independent set in $F = \text{Bor}(d, \theta)[X]$ containing both x^0 and x^2 . Hence, x^0 and x^2 are not adjacent in $F = \text{Bor}(d, \theta)[X]$, so $\text{dist}(x^0, x^2) \leq \pi - \theta$. However, as before, a simple proof by induction shows that $\text{dist}(x^0, x^2) \geq \pi - og(H)\iota$. This implies $\pi - og(H)\iota \leq \pi - \theta$, so $\iota \geq \frac{\theta}{og(H)}$, again a contradiction. Thus, $og(H') \geq og(H) + 4$.

By definition, we have $|V(H')| = |V(H)| \cdot |V(G)|$. Let $v \in V(H')$ and let $h \in V(H)$ such that $v \in V(G_h)$. Then,

$$d_{H'}(v) \geq d_{G_h}(v) \geq \left(\frac{1}{3} - \zeta\right) |V(G)| = \left(\frac{1}{3|V(H)|} - \frac{\zeta}{|V(H)|}\right) |V(H')| = \left(\frac{1}{3|V(H)|} - \varepsilon\right) |V(H')|.$$

Let J be a graph on at most N vertices such that there exists a homomorphism $\phi : V(H') \rightarrow V(J)$ from H' to J . Without loss of generality, let us assume that $V(J) = [N]$. Write $V(H) = \{h_1, \dots, h_{|V(H)|}\}$. For each $x \in X$, let $c(x) := (\phi(x_{h_1}), \dots, \phi(x_{h_{|V(H)|}})) \in [N]^{|V(H)|}$. Since we have $\chi(\text{Bor}(d, \iota)[X]) > N^{|V(H)|}$, it follows that c is not a proper coloring of $\text{Bor}(d, \iota)[X]$, so there exist $x, y \in X$ such that $xy \in E(\text{Bor}(d, \iota)[X])$ and $c(x) = c(y)$. We show that the map $\psi : V(H) \rightarrow [N], h \mapsto \phi(x_h)$ is a homomorphism from H to J . Let $h, h' \in V(H)$ such that $hh' \in E(H)$. Then, $x_h y_{h'} \in E(H')$ so $\phi(x_h) \phi(y_{h'})$ is an edge of J . By definition, we have $\psi(h) = \phi(x_h)$ and $\psi(h') = \phi(x_{h'}) = \phi(y_{h'})$ since $c(x) = c(y)$. Thus, $\psi(h) \psi(h') \in E(J)$, so ψ is indeed a homomorphism. $\square$

2.4 Proof of the lower bounds

We prove the following result, which, together with [Lemma 2.2](#), implies the lower bounds for [Theorems 1.2](#) and [1.3](#).

Proposition 2.9. *For every $k \geq 2$, for every $t \geq 3$, we have*

$$\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}) \geq \frac{1}{3|V(\Gamma_{k-2}^t)|}.$$

Proof. We show that for every $\varepsilon > 0$, and for every $N \geq 1$, there exists a graph G which is $\{C_3, C_5, \dots, C_{2k+1}\}$ -free, has minimum degree at least $\left(\frac{1}{3|V(\Gamma_{k-2}^t)|} - \varepsilon\right) |V(G)|$ and has no homomorphism to a $\{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}$ -free graph on at most N vertices. Given $\varepsilon > 0$ and $N \geq 1$, let G be the graph given by [Proposition 2.8](#) for $H = \Gamma_{k-2}^t$. Then, $og(G) \geq og(\Gamma_{k-2}^t) + 4 \geq 2k + 3$ by [Lemma 2.1](#), so G is indeed $\{C_3, C_5, \dots, C_{2k+1}\}$ -free and has minimum degree at least $\left(\frac{1}{3|V(\Gamma_{k-2}^t)|} - \varepsilon\right) |V(G)|$. By contradiction, suppose that G has a homomorphism to a $\{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}$ -free graph J on at most N vertices. By construction of G , we have that Γ_{k-2}^t has a homomorphism to J . Since $og(J) \geq 2k - 1$, [Lemma 2.7](#) implies that J contains Γ_{k-2}^t as a subgraph, a contradiction. $\square$

3 Existence of homomorphisms

We first need some preliminary results.

Lemma 3.1. *Let $\delta, \varepsilon > 0$ and let G be an n -vertex graph with minimum degree at least δn . There exists a set of at most $\lceil \log(1/\varepsilon)/\delta \rceil$ vertices of G which dominates all but at most εn vertices of G .*

Proof. Since G has minimum degree δn , a simple double-counting argument shows that for every set V' of k vertices of G , some vertex of G has at least δk neighbors in V' . Set $V_0 = V(G)$ and let $\ell = \lceil \log(1/\varepsilon)/\delta \rceil$. For every $i \in [\ell]$, let $v_i \in V(G)$ such that v_i has at least $\delta|V_{i-1}|$ neighbors in V_{i-1} , and let $V_i = V_{i-1} \setminus N(v_i)$. Then, $\{v_1, \dots, v_\ell\}$ dominates $V(G) \setminus V_\ell$ by definition. Furthermore, for every $i \in [\ell]$, we have $|V_i| \leq (1 - \delta)|V_{i-1}|$, so $|V_\ell| \leq (1 - \delta)^\ell n \leq e^{-\delta\ell} n \leq \varepsilon n$ by definition of ℓ . $\square$

We will use a slightly modified variant of the degree form of Szemerédi's Regularity Lemma [Sze78], see for instance [KS96] for a closely related statement. Recall that in a graph G , a pair (U, W) of disjoint nonempty subsets of $V(G)$ is ε -regular, for $\varepsilon > 0$, if for every $U' \subseteq U$ and $W' \subseteq W$ such that $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$, we have $|d(U, W) - d(U', W')| \leq \varepsilon$, where $d(U, W) = \frac{e(U, W)}{|U||W|}$.

Lemma 3.2. *For every $\varepsilon > 0$ and $M_0 \in \mathbb{N}$, there exists $M = M(\varepsilon, M_0)$ such that for any graph G , any partition $\mathcal{P}$ of $V(G)$ into at most M_0 pieces and any real number $\zeta \in [0, 1]$, there is a partition of $V(G)$ into sets $V_0, \dots, V_p$ with $p \leq M$ and a spanning subgraph G' of G with the following properties:*

- $|V_0| \leq \varepsilon|V(G)|$,
- For every $i \geq 1$, V_i is contained in a part of $\mathcal{P}$,
- $|V_i| = |V_j|$ for all $i, j \geq 1$,
- For every $v \in V(G)$, we have $d_{G'}(v) \geq d_G(v) - (\zeta + \varepsilon)|V(G)|$,
- V_i is an independent set in G' for all $i \geq 1$,
- In G' , for all $i \neq j \in [p]$, the pair (V_i, V_j) is ε -regular, with density either 0 or greater than ζ .

We will also use the following basic fact on regular pairs.

Lemma 3.3. *Let G be a graph, let $\varepsilon > 0$, let U, W be nonempty disjoint subsets of $V(G)$ such that the pair (U, W) is ε -regular with density greater than 2ε . Then, there are less than $\varepsilon|U|$ vertices $u \in U$ such that $|N_G(u) \cap W| < \varepsilon|W|$.*

Proof. Let $U' = \{u \in U : |N_G(u) \cap W| < \varepsilon|W|\}$. If $|U'| \geq \varepsilon|U|$, since the pair (U, W) is regular, we would have $d(U', W) \geq \varepsilon$, which would imply that some vertex $u \in U'$ has at least $\varepsilon|W|$ neighbors in W , a contradiction. $\square$

The following proposition yields the upper bounds of Theorems 1.2 and 1.3.

Proposition 3.4. *For every $k \geq 2$, for every $t \geq 3$, we have*

$$\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}) \leq \frac{1}{3|V(\Gamma_{k-2}^t)|}.$$

Before presenting the proof of Proposition 3.4 in detail, we give an overview of the idea in the special case of Theorem 1.1 (i.e. $k = 2, t = 3$). The proof can be seen as going in the opposite direction from the lower-bound construction. There, the graph is built from three pieces, each having minimum degree close to one third of its size, which leads to the threshold $1/9$. The upper bound follows by showing that any obstruction would give rise to an analogous three-part structure, which cannot fit once the relative minimum degree exceeds $1/9$.

We begin by finding a bounded set which dominates almost all vertices, and partition the dominated vertices so that any two vertices in the same part have a common neighbor. We then apply the Regularity Lemma, obtaining a bounded number of regularity classes and a cleaned subgraph $G' \subseteq G$ in which every pair of classes is either empty or regular of positive density, while decreasing each degree by only a small amount. For each vertex, we record approximately how many neighbors it has in each regularity class in G' , and group together vertices with the same profile. The odd girth assumption yields that every profile class is an independent set. Finally, we refine this partition once more so that short walks in the quotient can be lifted to short walks in G .

Suppose that the resulting quotient contains a triangle, with corresponding parts Q_1, Q_2, Q_3 . For each Q_j , consider its *support*, namely the set of regularity classes in which vertices of Q_j have positive density in G' . The fact that G contains no triangles or 5-cycles implies that these three supports are pairwise disjoint, and moreover that there are no edges in G' between any two classes belonging to

their union. For each j , consider also the set of regularity classes adjacent in G' to the support of Q_j . Again using the absence of short odd cycles, these three sets are pairwise disjoint. Since there are no edges between the supports, they are also disjoint from all three supports. Thus we obtain six pairwise disjoint sets of regularity classes.

The minimum degree condition, together with the odd girth assumption, implies that each support contains slightly more than $2/9$ of all regularity classes, while each of the three additional sets contains slightly more than $1/9$. Their total size is therefore greater than the total number of regularity classes, a contradiction.

This also explains the connection with the lower-bound construction: at the threshold $1/9$, three pieces of this type can fit together, while above $1/9$ the counting argument rules this out. The proof below carries out the same argument in the general setting of [Proposition 3.4](#). The main additional difficulty is that, in the general case, the relevant obstruction is no longer a triangle, and the contradictions with the odd-girth condition may involve walks of length up to $2k - 1$. We therefore iterate the refinement procedure sufficiently many times to ensure that all such short walks in the quotient can be lifted to G , inspired by similar arguments used in [\[ES20\]](#). Roughly speaking, the role of the three vertices of the triangle is then played by the s vertices of the corresponding Mycielskian: each gives a support of size slightly more than $2/(3s)$ and a disjoint neighborhood of size slightly more than $1/(3s)$, leading to the same counting contradiction. Again, this mirrors the structure of the lower-bound constructions, where each vertex of the obstruction gives rise to one such support-neighborhood pair.

Proof of [Proposition 3.4](#). Let $k \geq 2, t \geq 3$ and $\varepsilon > 0$. Let $s = |V(\Gamma_{k-2}^t)|$. Let G be an n -vertex $\{C_3, C_5, \dots, C_{2k+1}\}$ -free graph with minimum degree at least $(\frac{1}{3s} + \varepsilon)n$. We prove that G has a homomorphism to a $\{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}$ -free graph whose number of vertices only depends on k, t, ε .

Let $\eta := \varepsilon/7$. By [Lemma 3.1](#), there exists a set $D \subseteq V(G)$ of size at most $\gamma := \lceil 3s \log(1/\eta) \rceil$ which dominates all but at most $\eta \cdot n$ vertices of G . Let H be the subgraph of G induced by the vertices that are dominated by D and let $\mathcal{P}$ be a partition of $V(H)$ into at most γ parts such that any two vertices in a common part have a common neighbor: assign each dominated vertex to one chosen neighbor in D . Applying [Lemma 3.2](#) to H with $\varepsilon = \eta$ and $\zeta = 2\eta$ (and adding $V(G) \setminus V(H)$ to V_0 and keeping all the edges incident to these vertices), there exists some $M = M(\eta, \gamma) = M(k, t, \varepsilon)$ and a partition of $V(G)$ into sets $V_0, \dots, V_p$ and a spanning subgraph G' of G with the following properties:

- $p \leq M$,
- $|V_0| \leq 2\eta \cdot n$,
- For every $i \geq 1$, any two vertices in V_i have a common neighbor in G ,
- There exists $m \in \mathbb{N}$ such that $|V_i| = m$ for every $i \geq 1$,
- For every $v \in V(G)$, we have $d_{G'}(v) \geq (\frac{1}{3s} + \varepsilon - 3\eta)n$,
- V_i is an independent set in G' for all $i \geq 1$,
- In G' , for all $i \neq j \in [p]$, the pair (V_i, V_j) is η -regular, with density either 0 or greater than 2η .

For every $v \in V(G)$ and every $i \in [p]$, let $\mu_i(v) \in \{0, \eta, 2\eta, \dots, \lfloor 1/\eta \rfloor \eta\}$ such that $|N_{G'}(v) \cap V_i| \in [\mu_i(v) \cdot m, (\mu_i(v) + \eta) \cdot m]$. Informally, the vector $\mu(v)$ records the approximate density of v on each of the regularity classes $V_1, \dots, V_p$. Let $\mathcal{Q}$ be the partition of the vertices of G according to their vector $(\mu_1(v), \dots, \mu_p(v))$. Note that $\mathcal{Q}$ need not refine the previous partitions. For every part $Q \in \mathcal{Q}$, all vertices $v \in Q$ have the same vector $(\mu_1(v), \dots, \mu_p(v))$. We denote this vector by $\mu(Q) = (\mu_1(Q), \dots, \mu_p(Q))$. The *support* of Q is the set $\{i \in [p] : \mu_i(Q) > 0\}$, which we denote by $\text{supp}(Q)$. Observe that $\text{supp}(Q)$ corresponds to a set of regularity classes. Note also that M, γ and η are independent of n , so $|\mathcal{Q}|$ only depends on ε, k and t .

Claim 1. For every $Q \in \mathcal{Q}$, we have $\sum_{i \in [p]} \mu_i(Q) \geq (\frac{1}{3s} + \varepsilon - 6\eta)p$.

Proof of Claim 1. Let $v \in Q$, so $\sum_{i \in [p]} \mu_i(Q) = \sum_{i \in [p]} \mu_i(v)$. For every $i \in [p]$, by definition of $\mu_i(v)$ we have $\mu_i(v) \cdot m \geq |N_{G'}(v) \cap V_i| - \eta \cdot m$. Thus, using that $mp \leq n$, we have

$$\begin{aligned}
\sum_{i \in [p]} \mu_i(Q) &= \sum_{i \in [p]} \mu_i(v) \\
&\geq \sum_{i \in [p]} \left(\frac{|N_{G'}(v) \cap V_i|}{m} - \eta \right) \\
&= \left(\sum_{i \in [p]} \frac{|N_{G'}(v) \cap V_i|}{m} \right) - \eta p \\
&= \frac{1}{m} \sum_{i \in \{0, \dots, p\}} |N_{G'}(v) \cap V_i| - \frac{|N_{G'}(v) \cap V_0|}{m} - \eta p \\
&\geq \frac{|N_{G'}(v)|}{m} - \frac{|V_0|}{m} - \eta p \\
&\geq \left(\frac{1}{3s} + \varepsilon - 3\eta \right) \cdot \frac{n}{m} - 2\eta \cdot \frac{n}{m} - \eta p \\
&\geq \left(\frac{1}{3s} + \varepsilon - 6\eta \right) p.
\end{aligned}$$

◇

Claim 2. Each $Q \in \mathcal{Q}$ is an independent set in G .

Proof of Claim 2. By contradiction, suppose that some $Q \in \mathcal{Q}$ is not an independent set in G , and let $x, y \in Q$ be adjacent in G . By Claim 1, there exists $i \in [p]$ such that $\mu_i(Q) > 0$, so $\mu_i(x), \mu_i(y) > 0$. Thus, both x and y have a neighbor in V_i . However, any two vertices in V_i have a common neighbor in G , so G contains a closed walk of length 5, a contradiction since G has odd girth greater than $2k + 1 \geq 5$. ◇

Let $\mathcal{R}^1 = \mathcal{Q}$ and define partitions $\mathcal{R}^2, \dots, \mathcal{R}^k$ of $V(G)$ inductively as follows. Let $r \in \{2, \dots, k\}$ and suppose that $\mathcal{R}^{r-1}$ was defined. Then, let $\mathcal{R}^r$ be the partition of $V(G)$ where two vertices are in the same part of $\mathcal{R}^r$ if and only if they are in the same part of $\mathcal{R}^{r-1}$ and their neighbors in G belong to the same parts of $\mathcal{R}^{r-1}$. Note that $|\mathcal{R}^r| \leq |\mathcal{R}^{r-1}| \cdot 2^{|\mathcal{R}^{r-1}|}$, so $|\mathcal{R}^k|$ only depends on ε, k and t .

Let $G/\mathcal{R}^k$ be the graph on vertex set $\mathcal{R}^k$ where $R_i^k, R_j^k \in \mathcal{R}^k$ are adjacent if and only if there is an edge between R_i^k and R_j^k in G . Since $\mathcal{R}^k$ refines $\mathcal{Q}$, there is a natural homomorphism from G to $G/\mathcal{R}^k$ by Claim 2, and the size of $G/\mathcal{R}^k$ is upper bounded by a function of ε, t and k only. We now show that $G/\mathcal{R}^k$ is $\{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}$ -free, which will conclude the proof. By contradiction, suppose not. Then, there exist $R_1^k, \dots, R_s^k \in \mathcal{R}^k$ with the following properties:

- For all $i \in [s]$, there exists a closed walk of length $2k - 1$ in $G/\mathcal{R}^k$ that visits R_i^k ,
- For all $i \neq j \in [s]$, there exist an even walk and an odd walk between R_i^k and R_j^k in $G/\mathcal{R}^k$, both of length at most $2k - 2$.

Indeed, if $G/\mathcal{R}^k$ contains an odd cycle of length at most $2k - 3$, we can take all R_i^k equal to some part of $\mathcal{R}^k$ that belongs to such a cycle, and if $G/\mathcal{R}^k$ contains Γ_{k-2}^t , we can take the R_i^k as the vertices of a copy of Γ_{k-2}^t in $G/\mathcal{R}^k$. In the latter case, the two properties follow immediately from Lemma 2.6.

For every $r \in [k]$ and every $i \in [s]$, let R_i^r be the part of $\mathcal{R}^r$ that contains R_i^k . For convenience, let $Q_i = R_i^1$ for every $i \in [s]$.

Claim 3. For every $i \in [s]$, there exists a walk of length $2k - 1$ in G between two vertices of Q_i .

Proof of Claim 3. By symmetry, it suffices to prove it for $i = 1$. Let $R_1^k = S_1^k S_2^k \dots S_{2k-1}^k S_{2k}^k = R_1^k$ be a closed walk in $G/\mathcal{R}^k$, where $S_i^k \in \mathcal{R}^k$ for every $i \in [2k]$. For every $r \in [k]$ and every $i \in [2k]$, let S_i^r be the part of $\mathcal{R}^r$ that contains S_i^k . Since S_k^k and S_{k+1}^k are adjacent in $G/\mathcal{R}^k$, there exist $v_k \in S_k^k$ and $v_{k+1} \in S_{k+1}^k$ that are adjacent in G . Furthermore, there is an edge in G between S_k^k and S_{k-1}^k and between S_{k+1}^k and S_{k+2}^k . By definition of $\mathcal{R}^k$, this implies that v_k is adjacent in G to some vertex

$v_{k-1} \in S_{k-1}^{k-1}$ and v_{k+1} is adjacent in G to some vertex $v_{k+2} \in S_{k+2}^{k-1}$. After repeating this process $k-2$ extra times, we finally obtain a vertex $v_1 \in R_1^1$ and a vertex $v_{2k} \in R_1^1$. Then, $v_1, v_{2k} \in Q_1$ and they are connected by a walk of length $2k-1$ in G . $\diamond$

A similar proof shows that the following claim also holds.

Claim 4. For all $i \neq j \in [s]$, there exist an even walk of length at most $2k-2$ in G between a vertex of Q_i and a vertex of Q_j , and an odd walk of length at most $2k-2$ in G between a vertex of Q_i and a vertex of Q_j .

Claim 5. The sets $\text{supp}(Q_1), \dots, \text{supp}(Q_s)$ are pairwise disjoint.

Proof of Claim 5. By contradiction, suppose not: there exist $j \neq j' \in [s]$ and $i \in \text{supp}(Q_j) \cap \text{supp}(Q_{j'})$. By Claim 4, there exist $x_j \in Q_j$ and $x_{j'} \in Q_{j'}$ that are connected by a walk in G with an odd number of edges, and at most $2k-2$ edges (hence at most $2k-3$ edges). Since $i \in \text{supp}(Q_j) \cap \text{supp}(Q_{j'})$, we have $\mu_i(x_j), \mu_i(x_{j'}) > 0$ so both x_j and $x_{j'}$ have a neighbor in the regularity class V_i . However, any two vertices in V_i have a common neighbor, so x_j and $x_{j'}$ are connected by a walk in G with 4 edges. This proves the existence of a closed odd walk of length at most $2k+1$ in G , a contradiction. $\diamond$

Claim 6. For every $j \in [s]$, we have $|\text{supp}(Q_j)| \geq 2 \left(\frac{1}{3s} + \varepsilon - 6\eta \right) p$.

Proof of Claim 6. Let $j \in [s]$ and let $x_j, y_j \in Q_j$ be connected by a walk of length $2k-1$ in G ; such vertices exist by Claim 3. We first show that $\mu_i(Q_j) \leq 1/2$ for every $i \in [p]$. By contradiction, suppose not. Then, $\mu_i(x_j), \mu_i(y_j) > 1/2$ for some $i \in [p]$. Thus, $|N_{G'}(x_j) \cap V_i|, |N_{G'}(y_j) \cap V_i| > |V_i|/2$ so x_j and y_j have a common neighbor. Together with the walk of length $2k-1$ between them, this forms a closed walk of length $2k+1$ in G , so G contains a closed odd walk of length at most $2k+1$, a contradiction. This proves that $\mu_i(Q_j) \leq 1/2$ for every $i \in [p]$. Then, using Claim 1, we have

$$\begin{aligned} |\text{supp}(Q_j)| &\geq \sum_{i \in \text{supp}(Q_j)} 2\mu_i(Q_j) \\ &= 2 \sum_{i \in \text{supp}(Q_j)} \mu_i(Q_j) \\ &\geq 2 \left(\frac{1}{3s} + \varepsilon - 6\eta \right) p, \end{aligned}$$

as claimed. $\diamond$

Claim 7. If $i, i' \in \bigcup_{j \in [s]} \text{supp}(Q_j)$, there is no edge between V_i and $V_{i'}$ in G' .

Proof of Claim 7. For $i = i'$, this is immediate since V_i is an independent set in G' . Suppose now that $i \neq i'$, and suppose by contradiction that there is an edge between V_i and $V_{i'}$ in G' . Thus, the pair $(V_i, V_{i'})$ is η -regular in G' , with density greater than 2η . Consider first the case where i, i' are in the support of the same Q_j , and let $y \in Q_j$. Since $i, i' \in \text{supp}(Q_j)$, we have $\mu_i(y), \mu_{i'}(y) > 0$ so by definition y has at least $\eta \cdot m$ neighbors both in V_i and in $V_{i'}$. Since $(V_i, V_{i'})$ is η -regular in G' , with density greater than 2η , there is an edge in G' between $N_{G'}(y) \cap V_i$ and $N_{G'}(y) \cap V_{i'}$. This gives a triangle in G' , which is impossible. Consider now the case where $i \in \text{supp}(Q_j)$ and $i' \in \text{supp}(Q_{j'})$ with $j \neq j'$. By Claim 4, there exist $x_j \in Q_j$ and $x_{j'} \in Q_{j'}$ that are connected by a walk in G with an even number of edges, and at most $2k-2$ edges. By definition of μ , x_j has at least $\eta \cdot m$ neighbors in V_i and $x_{j'}$ has at least $\eta \cdot m$ neighbors in $V_{i'}$. Since $(V_i, V_{i'})$ is η -regular in G' , with density greater than 2η , there is an edge in G' between $N_{G'}(x_j) \cap V_i$ and $N_{G'}(x_{j'}) \cap V_{i'}$. Thus, x_j and $x_{j'}$ are connected by a walk in G with 3 edges. This proves the existence of a closed odd walk of length at most $2k+1$ in G , which is impossible. $\diamond$

For every $Q \in \mathcal{Q}$, let $\text{supp}^2(Q)$ be the set of indices $i \in [p]$ such that there is an edge in G' between V_i and $V_{i'}$ for some $i' \in \text{supp}(Q)$. Again, note that $\text{supp}^2(Q)$ corresponds to a set of regularity classes.

Claim 8. The sets $\text{supp}^2(Q_1), \dots, \text{supp}^2(Q_s)$ are pairwise disjoint.

Proof of Claim 8. Let $j \neq j' \in [s]$ and suppose by contradiction that there exists $\ell \in \text{supp}^2(Q_j) \cap \text{supp}^2(Q_{j'})$. Then, there exist $i \in \text{supp}(Q_j)$ and $i' \in \text{supp}(Q_{j'})$ such that there is an edge in G' between V_ℓ and V_i , and between V_ℓ and $V_{i'}$. By Claim 4, there exist $x_j \in Q_j$ and $x_{j'} \in Q_{j'}$ that are connected

by a walk in G with an odd number of edges, and at most $2k - 2$ edges (hence at most $2k - 3$ edges). Since $i \in \text{supp}(Q_j)$, we have $|N_{G'}(x_j) \cap V_i| \geq \eta \cdot m$. Similarly, we have $|N_{G'}(x_{j'}) \cap V_{i'}| \geq \eta \cdot m$. The pair $(V_i, V_{i'})$ is η -regular with density greater than 2η in G' so by [Lemma 3.3](#), the number of vertices $u \in V_i$ such that $|N_{G'}(u) \cap V_{i'}| < \eta|V_{i'}|$ is less than $\eta|V_i|$. Thus, there exists $z \in N_{G'}(x_j) \cap V_i$ such that $|N_{G'}(z) \cap V_{i'}| \geq \eta|V_{i'}|$. Since $(V_{i'}, V_{i'})$ is η -regular in G' , with density greater than 2η , there is an edge in G' between $N_{G'}(z) \cap V_{i'}$ and $N_{G'}(x_{j'}) \cap V_{i'}$. Thus, there is a walk of length 4 between x_j and $x_{j'}$ in G . This proves the existence of a closed odd walk of length at most $2k + 1$ in G , which is impossible. $\diamond$

Claim 9. For every $j \in [s]$, we have $|\text{supp}^2(Q_j)| \geq (\frac{1}{3s} + \varepsilon - 6\eta)p$.

Proof of L. et $j \in [s]$, let $i \in \text{supp}(Q_j)$ and let $v \in V_i$. Let $Q \in \mathcal{Q}$ such that $v \in Q$. It follows from [Claim 1](#) that $|\text{supp}(Q)| \geq (\frac{1}{3s} + \varepsilon - 6\eta)p$. However, if $i' \in \text{supp}(Q)$ then $\mu_{i'}(v) > 0$ so there is an edge between $v \in V_i$ and $V_{i'}$ in G' . Thus, $i' \in \text{supp}^2(Q_j)$. Therefore, $|\text{supp}^2(Q_j)| \geq (\frac{1}{3s} + \varepsilon - 6\eta)p$. $\diamond$

Claim 10. The sets $\text{supp}(Q_1), \dots, \text{supp}(Q_s), \text{supp}^2(Q_1), \dots, \text{supp}^2(Q_s)$ are pairwise disjoint.

Proof of Claim 10. By [Claim 5](#) and [Claim 8](#), it suffices to show that $\text{supp}(Q_j)$ and $\text{supp}^2(Q_{j'})$ are disjoint for all $j, j' \in [s]$. Let $i' \in \text{supp}^2(Q_1) \cup \dots \cup \text{supp}^2(Q_s)$. There exists $i \in \text{supp}(Q_1) \cup \dots \cup \text{supp}(Q_s)$ such that there is an edge between V_i and $V_{i'}$ in G' (hence in G), so $i' \notin \text{supp}(Q_1) \cup \dots \cup \text{supp}(Q_s)$ by [Claim 7](#). $\diamond$

By [Claims 6, 9](#) and [10](#), we then have

$$\left| \bigcup_{j \in [s]} \text{supp}(Q_j) \cup \text{supp}^2(Q_j) \right| = \sum_{j \in [s]} |\text{supp}(Q_j)| + |\text{supp}^2(Q_j)| \geq s \cdot 3 \cdot \left(\frac{1}{3s} + \varepsilon - 6\eta \right) p > p,$$

a contradiction. $\square$

For completeness, we briefly deduce the statements of [Theorems 1.1–1.3](#) from the preceding results.

Proof of Theorems 1.1–1.3. By [Propositions 3.4](#) and [2.9](#) we have that

$$\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}) = \frac{1}{3|V(\Gamma_{k-2}^t)|}$$

for every $k \geq 2$ and $t \geq 3$.

Observe that by the definition of the generalized Mycielskians, it follows that $\Gamma_0^t = K_t$ for every $t \geq 2$. Hence, plugging $k = 2$ into the above we immediately find that

$$\delta_{\text{hom}}(\{C_3, C_5\}, \{K_t\}) = \frac{1}{3t}$$

for every $t \geq 3$. This establishes the statement of [Theorem 1.2](#), and hence also [Theorem 1.1](#).

Now, for [Theorem 1.3](#), let $k \geq 2$ and a graph H of odd girth at least $2k - 1$ (and hence greater than $2(k-2)+1$) be given. Then by applying [Lemma 2.2](#) to the graph H and with parameter $k-2 \geq 0$ in place of k , we find that there exists some $t \geq 3$ such that H is a subgraph of Γ_{k-2}^t . Hence, any bounded-size $\{C_3, \dots, C_{2k-3}, H\}$ -free graph is automatically also a bounded-size $\{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}$ -free graph. It follows that

$$\begin{aligned} \delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, H\}) &\geq \delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}, \{C_3, \dots, C_{2k-3}, \Gamma_{k-2}^t\}) \\ &= \frac{1}{3|V(\Gamma_{k-2}^t)|} \\ &> 0. \end{aligned}$$

This establishes also the statement of [Theorem 1.3](#), concluding the proof. $\square$

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